Functions with Prescribed Quasisymmetry Quotients

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Let f be a continuous, increasing function of $\mathbf{R}$ into itself. Let

$$k_f(x, h) = \frac{f(x + h) - f(x)}{f(x) - f(x - h)}, \quad x \in \mathbf{R}, h > 0.$$

Observe that if $g = af + b$, $a, b \in \mathbf{R}$ then $k_g = k_f$. One can show conversely that if $k_g = k_f$ then $g = f$ up to a real affine transformation. The quantity k_f is called the *quasisymmetry quotient* of f. A function is *c-quasisymmetric*, $c \geq 1$, if

$$\text{(1)} \qquad \frac{1}{c} \leq k_f(x, h) \leq c.$$

The condition (1) also implies that $f(\pm\infty) = \pm\infty$. We refer to the book by Lehto and Virtanen [?] for a thorough discussion of the role of quasisymmetric functions in the theory of quasiconformal mappings.

Due to its simple form the quasisymmetry quotient necessarily satisfies any number of algebraic identities. Two of these are sufficient, as we shall prove in the following existence theorem.

Theorem *Suppose that $k(x, h)$ is a positive, continuous function on the open upper half-plane $\{(x, h): x \in \mathbf{R}, h > 0\}$ satisfying the identities*

$$\text{(2)} \qquad \frac{k(x, 2h)}{k(x, h)} = \frac{1 + k(x + h, h)}{1 + k(x - h, h)^{-1}},$$

and

$$\text{(3)} \qquad \frac{k(x, 3h)}{k(x, h)} = \frac{1 + k(x + h, h) + k(x + h, h)k(x + 2h, h)}{1 + k(x - h, h)^{-1} + k(x - h, h)^{-1}k(x - 2h, h)^{-1}}.$$

Then there is a continuous, increasing function f such that $k_f(x, h) = k(x, h)$.

If in addition $k(x, h)$ is bounded between $1/c$ and c, then of course f is c-quasisymmetric.

Proof. For all nonnegative integers m and n, let

$$f\left(\frac{m}{2^n}\right) = \frac{\sum_{r=0}^{m-1}\prod_{j=1}^{r} k(j/2^n, 1/2^n)}{\sum_{r=0}^{2^n-1}\prod_{j=1}^{r} k(j/2^n, 1/2^n)},$$

$$f\left(-\frac{m}{2^n}\right) = -\frac{\sum_{r=1}^{m}\prod_{j=0}^{r-1} k(-j/2^n, 1/2^n)^{-1}}{\sum_{r=0}^{2^n-1}\prod_{j=1}^{r} k(j/2^n, 1/2^n)}.$$

Here we follow the usual convention that an empty sum has the value 0 and an empty product has the value 1. Thus $f(0) = 0$ and $f(1) = 1$.

Equation (2) implies that f is well defined. To check this it is useful to establish, by induction, the following formulas:

$$\prod_{j=1}^{r} k(j/2^n, 1/2^n) = \frac{\prod_{j=1}^{2r} k(j/2^{n+1}, 1/2^{n+1}) + \prod_{j=1}^{2r+1} k(j/2^{n+1}, 1/2^{n+1})}{1 + k(1/2^{n+1}, 1/2^{n+1})},$$

$$\prod_{j=0}^{r-1} k(-j/2^n, 1/2^n)^{-1} = \frac{\prod_{j=0}^{2r-2} k(-j/2^{n+1}, 1/2^{n+1})^{-1} + \prod_{j=0}^{2r-1} k(-j/2^{n+1}, 1/2^{n+1})^{-1}}{1 + k(1/2^{n+1}, 1/2^{n+1})}.$$

Sum the first identity from $r = 0$ to $m - 1$ and then from $r = 0$ to $2^n - 1$. Dividing the results gives that $f(m/2^n) = f(2m/2^{n+1})$. Next sum the second identity from $r = 1$ to m and the first identity from $r = 0$ to $2^n - 1$. Dividing the results this time shows that $f(-m/2^n) = f(-2m/2^{n+1})$. It follows that f is well defined. By construction,

(4) $$\frac{f((m+1)/2^n) - f(m/2^n)}{f(m/2^n) - f((m-1)/2^n)} = k(m/2^n, 1/2^n), \quad m, n \in \mathbf{Z}, \quad n \geq 0.$$

So far, f is defined at the dyadic rationals, $m/2^n$, and it is strictly increasing. Let x be a nondyadic number and define $f(x)$ to be the supremum of f on the set of dyadic rationals less than x. This defines f everywhere as a strictly increasing function.

Consider the set of points (x, h) in the upper half-plane where $k(x, h) = k_f(x, h)$. By (4) this set contains all points $(m/2^n, 1/2^n)$, where m and n are integers and $n \geq 0$.

By (2) it contains $(x, 2h)$ whenever it contains $(x - h, h)$, (x, h), and $(x + h, h)$. Hence it contains all points $(m/2^n, 2^p)$, where $m, n, p \in \mathbf{Z}$, $n \geq 0$. Likewise by (3) it then contains all points $(m/2^n, 2^p 3^q)$, where $p, q \in \mathbf{Z}$, and $q \geq 0$. Let $\mathcal{D}$ denote the set of dyadic rationals and let $\mathcal{T}$ denote the rationals of the form $2^p 3^q$, $p, q \in \mathbf{Z}$, $q \geq 0$. Then $\mathcal{D} \times \mathcal{T}$ is dense in the upper half-plane and

$$(5) \qquad k(x, h) = \frac{f(x + h) - f(x)}{f(x) - f(x - h)}, \quad (x, h) \in \mathcal{D} \times \mathcal{T}.$$

Once we know that f is continuous it will then also follow that $k(x, h) = k_f(x, h)$ for all $x \in \mathbf{R}$, $h > 0$.

Solving (5) for $f(x)$ yields

$$(6) \qquad f(x) = \frac{f(x + h) + k(x, h)f(x - h)}{1 + k(x, h)}, \quad (x, h) \in \mathcal{D} \times \mathcal{T}.$$

Letting h decrease to a positive limit then implies that

$$(7) \qquad f(x) = \frac{f^+(x + h) + k(x, h)f^-(x - h)}{1 + k(x, h)}, \quad x \in \mathcal{D}, \ h > 0,$$

where f^+ is the right-hand limit of f and f^- is the left-hand limit.

Let $x_0 \in \mathbf{R}$. Consider $x < x_0$ with $x \in \mathcal{D}$ and let $x + h = x_0 + 1$. If we take the limit in (7) as $x \uparrow x_0$ we obtain

$$f^-(x_0) = \frac{f^+(x_0 + 1) + k(x_0, 1)f^-(x_0 - 1)}{1 + k(x_0, 1)}.$$

On the other hand, consider $x > x_0$ with $x \in \mathcal{D}$ and let $x - h = x_0 - 1$. Now if we take the limit as $x \downarrow x_0$ we get

$$f^+(x_0) = \frac{f^+(x_0 + 1) + k(x_0, 1)f^-(x_0 - 1)}{1 + k(x_0, 1)}.$$

Thus $f^+(x_0) = f^-(x_0)$, and f is continuous at x_0. Since x_0 was arbitrary this proves that f is continuous on $\mathbf{R}$, and completes the proof of the theorem.

In closing we remark that it is unlikely a single condition such as (2) on a function $k(x, h)$ will be sufficient by itself for k to be a quasisymmetry quotient. For example, let $\varphi_h(x) = \varphi(x, h)$ be a continuous function in the upper half-plane such that $\varphi(x, 2h) = \varphi(x, h)$ and such that the 1-parameter family of functions $x \mapsto \varphi_h(x)$ are uniformly quasisymmetric. If

$$k(x, h) = \frac{\varphi(x + h, h) - \varphi(x, h)}{\varphi(x, h) - \varphi(x - h, h)},$$

then k satisfies (2), but it is not of the form k_f unless all the functions $x \mapsto \varphi_h(x)$ are equivalent under the group of operations $f \mapsto af + b$.